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# Supersymmetry, potentials with bound states at arbitrary energies and multi-soliton configurations

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**Abstract.** The connection between the algebra of supersymmetry and the inverse scattering method is used to construct one-dimensional potentials with any specified number of non-degenerate bound states at arbitrary energies. The reflection coefficient of the potential so constructed is related to the reflection coefficient of a reference potential which supports no bound states. It is shown that, by choosing the reference potential to be $V=0$, it is possible to construct reflectionless potentials with bound states at arbitrary energies. The relationship of this construction based on supersymmetry to other known constructions of reflectionless potentials is established. It is shown that the symmetric reflectionless potential may be expressed as a linear combination of the squares of the bound state eigenfunctions with coefficients related to the wavenumbers associated with the bound states.

## 1. Introduction

It was first shown by Witten (1981) that the algebra of supersymmetry may be used to pair together two related Hamiltonians to construct supersymmetric quantum mechanics. Supersymmetric quantum mechanics is the study of the properties of Hamiltonians linked by the algebra of supersymmetry. The existence of a conserved supercharge associated with supersymmetry in supersymmetric quantum mechanics leads to the feature that the spectral properties of the members that form the supersymmetric pair are related to each other. This feature has aroused enormous interest in supersymmetric quantum mechanics. A variety of physical systems have been analysed using the concept of supersymmetry. It has been shown, for example, that the spectrum of the Dirac equation for a charged particle in a central Coulomb field can be explained simply by using the concept of supersymmetry (Sukumar 1985b). The level degeneracies of a Dirac electron in a constant magnetic field have been interpreted using supersymmetry (Khare and Maharana 1984, Blockley and Stedman 1985). The supersymmetry of the Dirac electron in the field of an electric monopole has been studied by d'Hoker and Vinet (1984) and Yamagishi (1984). The 'accidental' degeneracy of certain systems with spin-orbit coupling has been discussed using supersymmetry by Ui (1984), Balantekin (1985) and Niemi (1985). Kostelecky and Nieto (1984), Bernstein and Brown (1984) and Andrianov *et al* (1984) have discussed other applications of supersymmetry in atomic, nuclear and solid state physics. These are just a few examples of the variety of systems that have been analysed using supersymmetry.

It has been shown (Andrianov *et al* 1984, Sukumar 1985a, c) that the simplest non-trivial realisation of the algebra of supersymmetry leads to the result that every

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one-dimensional non-relativistic Hamiltonian $H$ can have a partner $\tilde{H}$ such that either (i) $\tilde{H}$ has the same set of eigenvalues as $H$ except for missing the ground state of $H$, (ii) $H$ has the same set of eigenvalues as $\tilde{H}$ except for missing the ground state of $\tilde{H}$ or (iii) $H$ and $\tilde{H}$ have identical spectra of eigenvalues. This result has been shown to be very useful in the study of inverse scattering (Sukumar 1985d). The inverse scattering method (Gelfand and Levitan 1955) provides a recipe for constructing potentials starting from the specific spectral features of a system. The concept of a supersymmetric partner can be used as the building block to construct the edifice of the inverse scattering method in a simple step by step procedure. Starting from a reference potential of known spectral features it is possible to construct a supersymmetric partner which differs from the reference system by the presence of a specific additional spectral feature (Sukumar 1985d). By repeating this procedure it is possible to construct potentials with any desired spectral features. In this paper it is shown that the algebra of supersymmetry may be used to construct potentials in one dimension which support any number of bound states at any specified energies.

The plan of the paper is as follows: § 2 provides a summary of the method discussed by Sukumar (1985c) for introducing an additional bound state to a given spectrum of a reference potential using the algebra of supersymmetry. Section 3 illustrates the procedure by constructing potentials with one and two bound states. Section 4 generalises this procedure to construct potentials with any number $n$ of bound states at arbitrary energies $E_i$, $i = 1, 2, \ldots, n$. It is also shown that reflectionless potentials with $n$ bound states may be constructed by choosing the reference potential to be $V = 0$. It is shown that the symmetric reflectionless potentials form a subset of a class of reflectionless potentials.

An algorithm for constructing symmetric reflectionless potentials with bound states at arbitrary energies was given many years ago by Kay and Moses (1956). This algorithm is known to be related to the algorithm for constructing multi-soliton solutions of the Korteweg-deVries equation (Gardner *et al* 1967, Scott *et al* 1973). The confining potentials of quark-antiquark systems have been constructed phenomenologically using the multi-soliton algorithm (Thacker *et al* 1978, Quigg *et al* 1980). In § 5 of this paper the relationship between the different representations of symmetric reflectionless potentials is established. Section 6 contains the conclusions.

## 2. Introduction of an additional bound state

Let $V(x)$, $-\infty \leqslant x \leqslant \infty$, be a potential that supports bound states at energies $E_m = -\gamma_m^2/2\mu$ where $\mu$ is the reduced mass. The Hamiltonian is given by

$$H = -\frac{1}{2\mu}\frac{d^2}{dx^2} + V(x). \tag{1}$$

Let $R(k)$ be the reflection coefficient for positive energies $E = k^2/2\mu$. The procedure for finding a supersymmetric partner to $H$ denoted by $\tilde{H}$, whose eigenvalue spectrum consists of all the eigenvalues $E_m$ and in addition a ground state eigenvalue, $\tilde{E} < E_m$, has already been given (Sukumar 1985c). An outline of the procedure is given below. The energy $\tilde{E} = -\tilde{\gamma}^2/2\mu$ lies below the ground state of $H$ and is not one of the eigenstates of $H$. Hence the two linearly independent solutions of the Schrödinger equation for the potential $V$ at energy $\tilde{E}$, denoted by $\varphi(\tilde{E})$ and $\xi(\tilde{E})$, are both non-normalisable. However, the two linearly independent solutions may always be

linearly superposed to produce a solution $\psi(\tilde{E})$ such that $\psi(\tilde{E})$ is nodeless in $-\infty < x < \infty$. Let

$$\psi(\tilde{E}) = \varphi(\tilde{E}) + \alpha\xi(\tilde{E}). \tag{2}$$

For a certain range of values of $\alpha$ denoted by $\mathcal{R}(\alpha)$ the linear combination of $\varphi(\tilde{E})$ and $\xi(\tilde{E})$ will remain nodeless. This in turn implies that there is a family of nodeless solutions $\psi(\tilde{E})$ corresponding to the range of values of $\alpha$ in $\mathcal{R}(\alpha)$. For the sake of notational convenience this dependence on $\alpha$ will not always be explicitly indicated. Furthermore it is clear that since $\psi(\tilde{E})$ grows at least as fast as $\exp(\tilde{\gamma}x)$ as $|x| \to \infty$, the nodelessness of $\psi(\tilde{E})$ will guarantee that $[\psi(\tilde{E})]^{-1}$ is normalisable. In terms of the non-normalisable but nodeless solution $\psi(\tilde{E})$, $H$ may be factorised as

$$H = A^+(\tilde{E})A^-(\tilde{E}) + \tilde{E} \tag{3}$$

where

$$A^{\pm}(\tilde{E}) = \frac{1}{\sqrt{2\mu}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right) \tag{4}$$

and

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(\tilde{E}). \tag{5}$$

$H$ has a supersymmetric partner $\tilde{H}$ given by

$$\tilde{H} = A^-(\tilde{E})A^+(\tilde{E}) + \tilde{E} \tag{6}$$

such that $(H - \tilde{E})$ and $(\tilde{H} - \tilde{E})$ are the diagonal elements of a supersymmetric Hamiltonian given by the anticommutator

$$\mathcal{H} = \{Q, Q^\dagger\} \tag{7}$$

where

$$Q = \begin{bmatrix} 0 & 0 \\ A^-(\tilde{E}) & 0 \end{bmatrix} \qquad Q^\dagger = \begin{bmatrix} 0 & A^+(\tilde{E}) \\ 0 & 0 \end{bmatrix}. \tag{8}$$

Furthermore

$$[Q, \mathcal{H}] = 0 = [Q^\dagger, \mathcal{H}]. \tag{9}$$

The existence of a conserved charge in supersymmetric systems leads to the general result that the partners of a supersymmetric pair have identical spectra except when the ground state of one member of the pair is annihilated by a charge operator. In the case of the supersymmetric system defined by (3)-(8), as shown in Sukumar (1985c), the solution of the Schrödinger equation for $\tilde{H}$ at energy $\tilde{E}$, denoted by $\tilde{\psi}(\tilde{E})$, is given by

$$\tilde{\psi}(\tilde{E}) = \frac{1}{\psi(\tilde{E})}. \tag{10}$$

$\tilde{\psi}(\tilde{E})$ is indeed the solution of

$$A^+(\tilde{E})\tilde{\psi}(\varepsilon) = 0 \qquad \varepsilon = \tilde{E}. \tag{11}$$

Since $[\psi(\tilde{E})]^{-1}$ is a nodeless normalisable function by construction, $\tilde{\psi}(\tilde{E})$ is the ground state of $\tilde{H}$ with eigenvalue $\tilde{E}$. $A^+(\tilde{E})$ is the operator that annihilates the ground state of $\tilde{H}$. All other eigenvalues of $\tilde{H}$ are also eigenvalues of $H$ and the solutions of the Schrödinger equations for $\tilde{H}$ and $H$ at a common energy $\varepsilon$ are related by

$$\tilde{\psi}(\varepsilon) \sim A^-(\tilde{E})\psi(\varepsilon) \tag{12a}$$

$$\psi(\varepsilon) \sim A^+(\tilde{E})\tilde{\psi}(\varepsilon) \qquad \varepsilon \neq \tilde{E}. \tag{12b}$$

These intertwining relations between the solutions $\psi$ and $\tilde{\psi}$ are valid not only when $\varepsilon$ is one of the common discrete eigenvalues $E_m$ but also when $\varepsilon$ is positive and when $\varepsilon$ is negative, but $\varepsilon \neq \tilde{E}$ and $\varepsilon \neq E_m$. (12) may be used to obtain a relation between the reflection coefficients of the potential $V$ and the potential corresponding to $\tilde{H}$ given by

$$\tilde{V} = V - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \psi(\tilde{E}). \tag{13}$$

For positive energies $E = k^2/2\mu$ the boundary conditions

$$\lim_{x\to-\infty} \psi(x, E) \sim \mathrm{e}^{\mathrm{i}kx} + R(k)\,\mathrm{e}^{-\mathrm{i}kx} \tag{14a}$$

$$\lim_{x\to+\infty} \psi(x, E) \sim T(k)\,\mathrm{e}^{\mathrm{i}kx} \tag{14b}$$

$$\lim_{x\to-\infty} \tilde{\psi}(x, E) \sim \mathrm{e}^{\mathrm{i}kx} + \tilde{R}(k)\,\mathrm{e}^{-\mathrm{i}kx} \tag{14c}$$

$$\lim_{x\to+\infty} \tilde{\psi}(x, E) \sim \tilde{T}(k)\,\mathrm{e}^{\mathrm{i}kx} \tag{14d}$$

when combined with (12) show that

$$\tilde{R}(k) = \frac{\tilde{\gamma} - \mathrm{i}k}{\tilde{\gamma} + \mathrm{i}k} R(k) \tag{15a}$$

$$\tilde{T}(k) = -\frac{\tilde{\gamma} - \mathrm{i}k}{\tilde{\gamma} + \mathrm{i}k} T(k). \tag{15b}$$

Thus starting from a potential $V$ with bound states at energies $E_m$ and positive energy reflection and transmission coefficients $R(k)$ and $T(k)$, it is possible to construct a potential $\tilde{V}$ given by (13) which supports bound states at energies $\tilde{E}$ and $E_m$. The ground state eigenfunction of $\tilde{V}$ is given by (10) while the remaining eigenfunctions are given by (12) for $\varepsilon = E_m$. The reflection and transmission coefficients are given by (15).

## 3. Potentials

### *3.1. Potentials with a single bound state*

Let $V_0(x)$ be a potential that supports no bound states and $R_0(k)$ be the reflection coefficient for positive energies. Using the procedure outlined in § 2 it is possible to find a potential $V_1$ which supports a single bound state at energy $E_1 = -\gamma_1^2/2\mu$. Using (10), $V_1$ may be written in the form

$$V_1 = V_0 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \psi_0(E_1) \tag{16}$$

where $\psi_0(E_1)$ is a nodeless unnormalisable solution of the Schrödinger equation for the potential $V_0$ at energy $E_1$. (10) and (12) show that the eigenfunction for the ground state energy $E_1$ is given by

$$\psi_1(E_1) \sim \frac{1}{\psi_0(E_1)} \tag{17}$$

while for $E \neq E_1$

$$\psi_1(E) \sim A_0^-(E_1)\psi_0(E) \tag{18}$$

where

$$A_0^-(E_1) = \frac{1}{\sqrt{2\mu}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln \psi_0(E_1)\right)\right]. \tag{19}$$

The reflection coefficient of $V_1$ is given by

$$R_1(k) = \frac{\gamma_1 - \mathrm{i}k}{\gamma_1 + \mathrm{i}k} R_0(k). \tag{20}$$

The above results may be illustrated by choosing $V_0 = 0$. $H_0$ is then the free particle Hamiltonian and $R_0(k) = 0$. (20) shows that the reflection coefficient of the supersymmetric partner $H_1$ also vanishes identically. It is clear that

$$\psi_{0R}(E_1) = \cosh \gamma_1 x + \alpha_1 \sinh \gamma_1 x \qquad |\alpha_1| < 1. \tag{21}$$

The condition $|\alpha_1| < 1$ ensures that $\psi_0(E_1)$ is nodeless although non-normalisable. The suffix R is added to indicate the reflectionless case. The reflectionless potential with a single bound state at $E_1$ is given by

$$V_{1R} = -(\gamma_1^2/\mu)\,\mathrm{sech}^2(\gamma_1 x + \tanh^{-1}\alpha_1) \tag{22}$$

and the ground state eigenfunction is given by

$$\psi_{1R}(E_1) \sim \mathrm{sech}(\gamma_1 x + \tanh^{-1}\alpha_1). \tag{23}$$

For $\alpha_1 = 0$, $V_{1R}$ is a symmetric potential. Using the suffix S to indicate 'symmetric'

$$V_{1SR} = -(\gamma_1^2/\mu)\,\mathrm{sech}^2\,\gamma_1 x \tag{24}$$

is the symmetric reflectionless potential with a single bound state at energy $E_1$. Normalised eigenfunctions will hereafter be denoted by the addition of a tilde. In terms of the normalised ground state eigenfunction given by

$$\tilde{\psi}_{1SR}(E_1) = (\gamma_1/2)^{1/2}\,\mathrm{sech}\,\gamma_1 x \tag{25}$$

the potential may be written in the form

$$V_{1SR} = -2(\gamma_1/\mu)\tilde{\psi}_{1SR}^2(E_1). \tag{26}$$

### *3.2. Potentials with two bound states*

The procedure used in the previous section may be repeated to find a potential with two bound states at energies $E_1$ and $E_2 < E_1$. Using (10), $V_2$ is given by

$$V_2 = V_1 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \psi_1(E_2) \tag{27}$$

where $\psi_1(E_2)$ is the nodeless non-normalisable solution of the Schrödinger equation for the potential $V_1$ at energy $E_2 = -\gamma_2^2/2\mu$. (10) and (12) show that the ground state eigenfunction is given by

$$\psi_2(E_2) \sim \frac{1}{\psi_1(E_2)} \tag{28}$$

while the eigenfunctions for other energies are given by

$$\psi_2(E) \sim A_1^-(E_2)\psi_1(E) \qquad E \neq E_2 \tag{29}$$

where

$$A_1^-(E_2) = \frac{1}{\sqrt{2\mu}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln \psi_1(E_2)\right)\right]. \tag{30}$$

In particular the first excited state of $V_2$ at energy $E_1$ has the eigenfunction

$$\psi_2(E_1) \sim A_1^-(E_2)\psi_1(E_1). \tag{31}$$

Using (16) the potential $V_2$ may be written in the form

$$V_2 = V_0 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln[\psi_0(E_1)\psi_1(E_2)]. \tag{32}$$

The reflection coefficient of $V_2$ for positive energies is given, using (15) and (20), by

$$R_2(k) = \frac{\gamma_2 - \mathrm{i}k}{\gamma_2 + \mathrm{i}k}\frac{\gamma_1 - \mathrm{i}k}{\gamma_1 + \mathrm{i}k}R_0(k). \tag{33}$$

The above expressions for $V_2$ and $\psi_2$ are given in terms of the solution $\psi_1$. It would be more convenient to express all quantities in terms of solutions in the reference potential $V_0$ which has no bound states. (18) and (19) show that

$$\psi_1(E_2) \sim \left[-\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln \psi_0(E_1)\right)\right]\psi_0(E_2). \tag{34}$$

Hence

$$\psi_0(E_1)\psi(E_2) \sim \det D_2 \tag{35}$$

where $D_2$ is a $2\times 2$ matrix given by

$$D_2 = \begin{bmatrix} \psi_0(E_1) & \psi_0(E_2) \\ \dot{\psi}_0(E_1) & \dot{\psi}_0(E_2) \end{bmatrix}. \tag{36}$$

These expressions may be used to write the potential with two bound states in the form

$$V_2 = V_0 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \det D_2. \tag{37}$$

The ground state eigenfunction is given by

$$\psi_2(E_2) \sim \frac{\psi_0(E_1)}{\det D_2} \sim [D_2^{-1}]_{22}. \tag{38}$$

The eigenfunction of the first excited state may be simplified to the form

$$\psi_2(E_1) \sim \frac{\psi_0(E_2)}{\det D_2} \sim [D_2^{-1}]_{12}. \tag{39}$$

Thus the potential is expressed in terms of the second derivative of the determinant of $D_2$ while the unnormalised eigenfunctions are given by the elements in the last column of the inverse of the matrix $D_2$. The condition that $\psi_0(E_1)$ and $\psi_1(E_2)$ must be chosen to be nodeless is equivalent to the requirement that $\psi_0(E_1)$ and $\psi_0(E_2)$ must be chosen such that the determinant of $D_2$ is free of zeros.

To illustrate the above results the case $V_0 = 0$ may be considered as in § 3.1. When $V_0 = 0$ the reflection coefficient of the resulting potential with two bound states also vanishes as shown by (33) for $R_0(k) = 0$. It is easy to see that

$$\psi_0(E_1) = \cosh \gamma_1 x + \alpha_1 \sinh \gamma_1 x \tag{40}$$

$$\psi_0(E_2) = \sinh \gamma_2 x + \alpha_2 \cosh \gamma_2 x. \tag{41}$$

The condition that det $D_2$ be free of zeros can be met only if $|\alpha_1| < 1$ and $|\alpha_2| < 1$. The symmetric reflectionless potential with bound states at $E_1$ and $E_2$, obtained by choosing $\alpha_1 = 0$ and $\alpha_2 = 0$, is given by

$$V_{2SR} = -\frac{1}{\mu}\frac{d^2}{dx^2} \ln \det D_{2SR} \tag{42}$$

where

$$D_{2SR} = \begin{bmatrix} \cosh \gamma_1 x & \sinh \gamma_2 x \\ \gamma_1 \sinh \gamma_1 x & \gamma_2 \cosh \gamma_2 x \end{bmatrix}. \tag{43}$$

The potential may be reduced to the form

$$V_{2SR} = -\frac{\gamma_2^2 - \gamma_1^2}{\mu} \frac{\gamma_2^2 \cosh^2 \gamma_1 x + \gamma_1^2 \sinh^2 \gamma_2 x}{(\gamma_2 \cosh \gamma_2 x \cosh \gamma_1 x - \gamma_1 \sinh \gamma_2 x \sinh \gamma_1 x)^2}. \tag{44}$$

Using (38) and (39) and the results in appendix 3 the normalised eigenfunctions may be written as

$$\tilde{\psi}_{2SR}(E_2) = \left(\frac{\gamma_2}{2}(\gamma_2^2 - \gamma_1^2)\right)^{1/2} \frac{\cosh \gamma_1 x}{\det D_{2SR}} \tag{45a}$$

$$\tilde{\psi}_{2SR}(E_1) = \left(\frac{\gamma_1}{2}(\gamma_2^2 - \gamma_1^2)\right)^{1/2} \frac{\sinh \gamma_2 x}{\det D_{2SR}}. \tag{45b}$$

In terms of these normalised eigenfunctions the symmetric reflectionless potential may be written in the form

$$V_{2SR} = -\frac{2}{\mu}[\gamma_2 \tilde{\psi}_{2SR}^2(E_2) + \gamma_1 \tilde{\psi}_{2SR}^2(E_1)]. \tag{46}$$

Certain features of $V_{2SR}$ may be readily established:

$$\lim_{x \to 0} V_{2SR}(x) = -\frac{\gamma_2^2 - \gamma_1^2}{\mu}[1 + (3\gamma_1^2 - \gamma_2^2)x^2 + \ldots] \tag{47a}$$

$$\lim_{x \to 0} \frac{d}{dx} V_{2SR}(x) = 0 \tag{47b}$$

$$\lim_{x \to 0} \frac{d^2}{dx^2} V_{2SR}(x) = -2(\gamma_2^2 - \gamma_1^2)(3\gamma_1^2 - \gamma_2^2) \tag{47c}$$

$$\lim_{|x| \to \infty} V_{2SR}(x) = 0. \tag{47d}$$

Analysis of these limits, together with the condition for the vanishing of $d/dx(V_{2SR})$, shows that

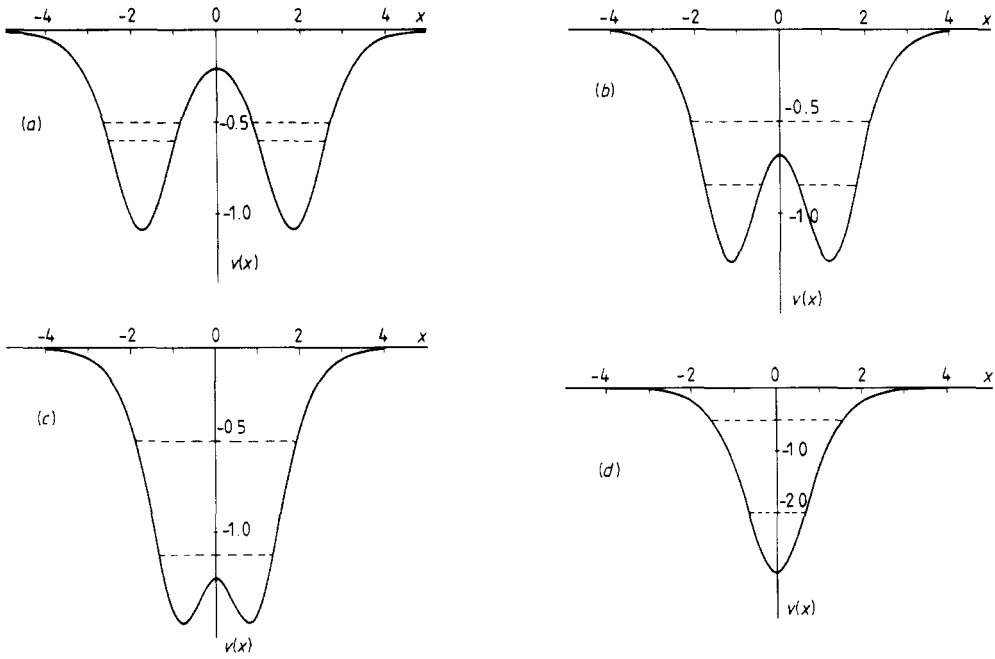


**Figure 1.** A symmetric reflectionless potential with bound states at energies $E_1 = -\gamma_1^2/2\mu$ and $E_2 = -\gamma_2^2/2\mu$ for $\mu = 1$, $\gamma_1 = 1$ and $\gamma_2 = (a)$ 1.1, $(b)$ 1.3, $(c)$ 1.5, $(d)$ 2.0. The locations of the bound levels are indicated by broken lines.

(i) if $\gamma_2^2 > 3\gamma_1^2$, $x = 0$ is a minimum of the potential and there are no additional minima. $V_{2SR}$ is then a symmetric single well. In particular if $\gamma_2^2 = 4\gamma_1^2$ the resulting potential is

$$V_{2SR} = -3(\gamma_1^2/\mu)\,\mathrm{sech}^2\,\gamma_1 x \tag{48}$$

i.e. $V_{2SR}$ is a $\mathrm{sech}^2 x$ potential with bound states at $-2\gamma_1^2/\mu$ and $-\gamma_1^2/2\mu$;

(ii) if $\gamma_2^2 < 3\gamma_1^2$, $x = 0$ is a maximum of the potential and there is a pair of additional minima for $|x| \neq 0$. $V_{2SR}$ under these conditions is a symmetric double well. If, furthermore, $\gamma_2^2 < 2\gamma_1^2$, then $V_{2SR}(x = 0) > E_2$ and at least the ground state lies inside the double well. If $\gamma_2^2 < \frac{3}{2}\gamma_1^2$ both the ground state and the first excited state lie inside the double well. These features are illustrated in figure 1. The above analysis shows that a class of symmetric reflectionless double well potentials with two bound states at specified energies $E_1$ and $E_2$, $E_2 < 3E_1$ may be simply defined.

## 4. Potentials with arbitrary number of bound states

By extension of the procedure outlined in §§ 2 and 3 it is possible to construct a hierarchy of Hamiltonians with successively increasing numbers of bound states starting from the Hamiltonian $H_0$ with no bound states. Denoting the Hamiltonian with $n$ bound states by $H_n$ and the ground state energy of $H_n$ by $E_n$:

$$E_n = -\frac{1}{2\mu}\gamma_n^2 \qquad \gamma_n^2 > \gamma_{n-1}^2 > \ldots > \gamma_1^2 \tag{49}$$

the Hamiltonian hierarchy is given by

$$H_m = A^-_{m-1}(E_m)A^+_{m-1}(E_m) + E_m = H_{m-1} + [A^-_{m-1}(E_m), A^+_{m-1}(E_m)]$$
$$m = 1, 2, \ldots, n \tag{50}$$

where

$$A_{m-1}(E_m) = \frac{1}{\sqrt{2\mu}}\left[\pm\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_{m-1}(E_m)\right)\right] \tag{51}$$

and $\psi_{m-1}(E_m)$ is a non-normalisable nodeless solution of the eigenvalue equation for $H_{m-1}$ at energy $E_m$ which lies below the ground state of $H_{m-1}$. The potentials in the hierarchy are related by

$$V_m = V_{m-1} - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_{m-1}(E_m). \tag{52}$$

The ground state eigenfunction of $H_m$ is given by

$$\psi_m(E_m) \sim 1/\psi_{m-1}(E_m) \tag{53}$$

while all the other eigenfunctions of $H_m$ are given in terms of the eigenfunctions of $H_{m-1}$ by

$$\psi_m(E_i) \sim A^-_{m-1}(E_m)\psi_{m-1}(E_i) \qquad i = 1, 2, \ldots, m-1;\ m = 1, 2, \ldots, n. \tag{54}$$

This network of interrelated eigenfunctions can be disentangled to express all eigenfunctions in terms of the solutions in the reference potential $V_0$. Iteration of (52) shows that the potential with $n$ bound states is related to $V_0$ by

$$V_n = V_0 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}[\ln\psi_0(E_1)\psi_1(E_2)\ldots\psi_{n-1}(E_n)]. \tag{55}$$

Using appendix 2, the product of wavefunctions in the above equation may be expressed in terms of the solutions $\psi_0(E_i)$ in the potential $V_0$ for various energies $E_i$. It is then possible to express $V_n$ in the form

$$V_n = V_0 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\det D_n \tag{56}$$

where the $n \times n$ matrix $D_n$ is given by

$$[D_n]_{JK} = \frac{\mathrm{d}^{J-1}}{\mathrm{d}x^{J-1}}\psi_0(E_K) \qquad J, K = 1, 2, \ldots, n. \tag{57}$$

The eigenfunctions of the potential $V_n$ may be expressed in the form

$$\psi_n(E_i) \sim [D_n^{-1}]_{in} \qquad i = 1, 2, \ldots, n. \tag{58}$$

The proof that the elements in the column $n$ of the inverse of the matrix $D_n$ are indeed the eigenfunctions of the potential $V_n$ in (56) with eigenenergies $E_i$, $i = 1, 2, \ldots, n$ is given in appendix 2. The requirement that $\psi_{m-1}(E_m)$, $m = 1, 2, \ldots, n$, be nodeless can be met by choosing the non-normalisable solutions $\psi_0(E_m)$, $m = 1, 2, \ldots, n$, such that $\det D_n$ has no zeros. The eigenfunction relation given in (54) may be extended to positive energies to show that the reflection coefficient of $V_m$ is related to the reflection coefficient of $V_{m-1}$ by

$$R_m(k) = \frac{\gamma_m - \mathrm{i}k}{\gamma_m + \mathrm{i}k}R_{m-1}(k). \tag{59}$$

Iteration of this relation gives

$$R_m(k) = \left[ \prod_{m=1,2,\ldots,n} \left( \frac{\gamma_m - ik}{\gamma_m + ik} \right) \right] R_0(k). \tag{60}$$

Equations (56), (57) and (58) provide a recipe for constructing potentials with bound states at specified energies $E_m$ and reflection coefficient for positive energies given by (60).

The algorithm for constructing reflectionless potentials with $n$ bound states is a particular case of the procedure given above corresponding to the choice $V_0 = 0$. Since $R_0(k) = 0$ when $V_0 = 0$, $R_n(k)$ also vanishes identically as shown by (60). The free particle solutions $\psi_{0R}$ at energies $E_J$ are given by

$$\psi_{0R}(E_{2J+1}) = \cosh \gamma_{2J+1} x + \alpha_{2J+1} \sinh \gamma_{2J+1} x$$

$$\psi_{0R}(E_{2J}) = \sinh \gamma_{2J} x + \alpha_{2J} \cosh \gamma_{2J} x \qquad J = 0, 1, \ldots, \leqslant \tfrac{1}{2}n. \tag{61}$$

It is easy to show that $\det D_n$ for this choice of $\psi_0$ has no zeros provided $|\alpha_m| < 1$, $m = 1, 2, \ldots, n$. A symmetric reflectionless potential with $n$ bound states may be obtained by choosing $\alpha_m = 0$, $m = 1, 2, \ldots, n$. The potential so obtained is given by

$$V_{nSR} = -\frac{1}{\mu} \frac{d^2}{dx^2} \ln \det D_{nSR} \tag{62}$$

where the elements of the matrix $D_{nSR}$ are given by

$$[D_{nSR}]_{JK} = \tfrac{1}{2}(\gamma_K)^{J-1}[e^{\gamma_K x} + (-1)^{J+K} e^{-\gamma_K x}]. \tag{63}$$

The unnormalised eigenfunctions are given in terms of the elements in the column $n$ of the inverse of the matrix $D_{nSR}$ by

$$\psi_{nSR}(E_i) \sim [D^{-1}_{nSR}]_{in} \qquad i = 1, 2, \ldots, n. \tag{64}$$

The normalisation of these eigenfunctions is discussed in appendix 3. The normalised eigenfunction for the eigenenergy $E_i$ may be written in the form

$$\tilde{\psi}_{nSR}(E_i) = \left( \frac{\gamma_i}{2} \prod_{K \neq i}^{n} |\gamma_K^2 - \gamma_i^2| \right)^{1/2} [D^{-1}_{nSR}]_{in} \qquad i = 1, 2, \ldots, n. \tag{65}$$

The relationship of the representation of the symmetric reflectionless potentials given in (56) and (57) to other seemingly different representations of the same potential is discussed in the next section.

## 5. Equivalence of reflectionless potentials

The matrix $D_{nSR}$ in (63) for the case of the symmetric reflectionless potential may be written in the form

$$D_{nSR} = \tfrac{1}{2}[A_1 + A_2] \tag{66}$$

where the elements of $A_1$ and $A_2$ are given by

$$[A_1]_{iJ} = \gamma_J^{i-1} e^{\gamma_J x} \tag{67}$$

$$[A_2]_{iJ} = (-1)^{i+J} \gamma_J^{i-1} e^{-\gamma_J x}. \tag{68}$$

The matrix $A_1$ can be easily inverted. The elements of $A_1^{-1}$ are given by

$$[A_1^{-1}]_{iJ} = (-1)^{i+J}\, e^{-\gamma_i x}\left(\prod_{K\neq i} |\gamma_K - \gamma_i|\right)^{-1} f_{iJ} \tag{69a}$$

where $f_{iJ}$ are coefficients in the expansion

$$\prod_{K\neq i} (y+\gamma_K) = \sum_{J=1}^{n} y^{J-1} f_{iJ}. \tag{69b}$$

By considering a diagonal matrix $G$ whose elements are related to the normalisation coefficients of the eigenfunctions discussed in appendix 3 and given by

$$G_{iJ} = \delta_{iJ}\left(\frac{\gamma_J}{2}\prod_{K\neq J} |\gamma_K^2 - \gamma_J^2|\right)^{1/2} \tag{70}$$

it can be shown after straightforward algebra that the matrix $M$ defined by

$$M = 2GA_1^{-1}D_{n\mathrm{SR}}G^{-1} \tag{71}$$

has elements

$$[M]_{JK} = \delta_{JK} + \frac{\lambda_J(x)\lambda_K(x)}{(\gamma_J+\gamma_K)} \tag{72}$$

where

$$\lambda_J(x) = C_j\, e^{-\gamma_J x} \tag{73}$$

and

$$\frac{C_J^2}{2\gamma_J} = \prod_{K\neq J}\left|\frac{\gamma_K+\gamma_J}{\gamma_K-\gamma_J}\right|. \tag{74}$$

The analysis of Kay and Moses (1956) and the $n$-soliton solution of the Korteweg-deVries equation (Gardner *et al* 1967, Scott *et al* 1973) lead to the result that the symmetric reflectionless potential may be expressed in terms of the matrix $M$ in the form

$$V = -\frac{1}{\mu}\frac{d^2}{dx^2}\ln\det M. \tag{75}$$

Since

$$\det M = 2(\det D_{n\mathrm{SR}})(\det A_1^{-1}) \tag{76}$$

and

$$\det A_1^{-1} \propto \exp\left(-\sum_i \gamma_i x\right) \tag{77}$$

it is clear that

$$\frac{d^2}{dx^2}\ln\det M = \frac{d^2}{dx^2}\ln\det D_{n\mathrm{SR}}. \tag{78}$$

This equality shows that the reflectionless potentials defined by (62) and (75) are identical.

It is shown in appendix 4 that the reflectionless potential with $n$ bound states may be expressed in terms of the normalised bound state eigenfunctions in the form

$$V_{n\mathrm{SR}} = -\frac{2}{\mu}\sum_{J=1}^{n}[\gamma_J \tilde{\psi}^2_{n\mathrm{SR}}(E_J)]. \tag{79}$$

Equation (79) is the generalised form of the result shown by (26) and (46) for the cases $n=1$ and $n=2$.

## 6. Conclusions

It has been shown that by repeatedly using the algebra of supersymmetry in a step by step fashion it is possible to construct potentials with bound states at arbitrary energies. It has been shown that the non-normalisable solutions in a reference potential which supports no bound states constitute the input in this construction. The reflection coefficient of the potential with $n$ bound states constructed by this procedure is related to the reflection coefficient of the reference potential. $V_n$ is in general not only a function of the $n$ bound state energies $E_i$ but also a function of $n$ parameters $\alpha_i$, $i=1,2,\ldots,n$. $\alpha_i$ characterises a particular linear superposition of the two linearly independent non-normalisable solutions in the reference potential $V_0$ at energy $E_i$. $\alpha_i$ can take such values that ensure that the determinant of $D_n$ is free of zeros. By choosing the reference potential to be $V_0=0$ reflectionless potentials with $n$ bound states may be constructed. The reflectionless potential so obtained is not necessarily a symmetric function of $x$. It has been shown that by choosing the parameters $\alpha_i$ to have specific values symmetric reflectionless potentials can be constructed. The resulting symmetric potential has been shown to be identical to the potential constructed using the $n$-soliton solution of the Korteweg-deVries equation. It has also been demonstrated that the symmetric reflectionless potential may be expressed in terms of the normalised bound state eigenfunctions in a particularly simple manner.

## Acknowledgment

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## Appendix 1

The Schrödinger equation for the Hamiltonian $H_n$ defined in (50) leads to the solution $\psi_n(E)$ for energy $E$. $\psi_n(E)$ is linked to the solution $\psi_{n-1}(E)$ of the Hamiltonian $H_{n-1}$ by (53) and (54). This wavefunction relation may be written in the form

$$\psi_n(E) \sim \psi_{n-1}(E_n)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi_{n-1}(E)}{\psi_{n-1}(E_n)}\right) \qquad E \neq E_n \tag{A1.1}$$

where $E_n$ is the ground state energy of $H_n$. Hence

$$\psi_{n-1}(E_n)\psi_n(E) \sim \det\begin{bmatrix}\psi_{n-1}(E_n) & \psi_{n-1}(E)\\ \dot{\psi}_{n-1}(E_n) & \dot{\psi}_{n-1}(E)\end{bmatrix}. \tag{A1.2}$$

Now let

$$F=\psi_{n-2}(E_{n-1})\psi_{n-1}(E_n)\psi_n(E)=\det\begin{bmatrix}\psi_{n-2}(E_{n-1}) & 0 & 0\\ 0 & \psi_{n-1}(E_n) & \psi_{n-1}(E)\\ 0 & \dot\psi_{n-1}(E_n) & \dot\psi_{n-1}(E)\end{bmatrix}. \tag{A1.3}$$

To express $F$ entirely in terms of the solutions of $H_{n-2}$, (54) can be used to express the solutions of $H_{n-1}$ in terms of the solutions of $H_{n-2}$:

$$\psi_{n-1}(\tilde E)\sim\left[-\frac{\mathrm d}{\mathrm dx}+\left(\frac{\mathrm d}{\mathrm dx}\ln\psi_{n-2}(E_{n-1})\right)\right]\psi_{n-2}(\tilde E)\qquad \tilde E\neq E_{n-1}. \tag{A1.4}$$

The first derivative of (A1.4) gives

$$\dot\psi_{n-1}(\tilde E)\sim-\ddot\psi_{n-2}(\tilde E)+\ddot\psi_{n-2}(E_{n-1})\frac{\psi_{n-2}(\tilde E)}{\psi_{n-2}(E_{n-1})}+\dot\psi_{n-2}(E_{n-1})\frac{\mathrm d}{\mathrm dx}\left(\frac{\psi_{n-2}(\tilde E)}{\psi_{n-2}(E_{n-1})}\right). \tag{A1.5}$$

Using the alternate expression of (A1.4) in the form

$$\psi_{n-1}(\tilde E)\sim\psi_{n-2}(E_{n-1})\frac{\mathrm d}{\mathrm dx}\left(\frac{\psi_{n-2}(\tilde E)}{\psi_{n-2}(E_{n-1})}\right)\qquad \tilde E\neq E_{n-1} \tag{A1.6}$$

it is possible to write

$$\dot\psi_{n-1}(\tilde E)+\alpha\psi_{n-1}(\tilde E)\sim-\ddot\psi_{n-2}(\tilde E)+\ddot\psi_{n-2}(E_{n-1})\frac{\psi_{n-2}(\tilde E)}{\psi_{n-2}(E_{n-1})} \tag{A1.7}$$

where

$$\alpha=-\frac{\dot\psi_{n-2}(E_{n-1})}{\psi_{n-2}(E_{n-1})}. \tag{A1.8}$$

Applying (A1.8) for $\tilde E=E_n$ and $\tilde E=E$ and using a matrix relation valid for any matrix $A$, namely

$$\det\begin{bmatrix}A_{11} & 0 & 0\\ 0 & A_{22} & A_{23}\\ 0 & A_{32} & A_{33}\end{bmatrix}=\det\begin{bmatrix}A_{11} & 0 & 0\\ 0 & A_{22} & A_{23}\\ 0 & A_{32}+\alpha A_{22} & A_{33}+\alpha A_{23}\end{bmatrix} \tag{A1.9}$$

it is then possible to write $F$ as

$$F=\det\begin{bmatrix}\psi_{n-2}(E_{n-1}) & 0 & 0\\ 0 & \dot\psi_{n-2}(E_n)-\dot\psi_{n-2}(E_{n-1})\dfrac{\psi_{n-2}(E_n)}{\psi_{n-2}(E_{n-1})} & \dot\psi_{n-2}(E)-\dot\psi_{n-2}(E_{n-1})\dfrac{\psi_{n-2}(E)}{\psi_{n-2}(E_{n-1})}\\ 0 & \ddot\psi_{n-2}(E_n)-\ddot\psi_{n-2}(E_{n-1})\dfrac{\psi_{n-2}(E_n)}{\psi_{n-2}(E_{n-1})} & \ddot\psi_{n-2}(E)-\ddot\psi_{n-2}(E_{n-1})\dfrac{\psi_{n-2}(E)}{\psi_{n-2}(E_{n-1})}\end{bmatrix}. \tag{A1.10}$$

Now consider

$$G=\det\begin{bmatrix}\psi_{n-2}(E_{n-1}) & \psi_{n-2}(E_n) & \psi_{n-2}(E)\\ \dot\psi_{n-2}(E_{n-1}) & \dot\psi_{n-2}(E_n) & \dot\psi_{n-2}(E)\\ \ddot\psi_{n-2}(E_{n-1}) & \ddot\psi_{n-2}(E_n) & \ddot\psi_{n-2}(E)\end{bmatrix}. \tag{A1.11}$$

For any matrix $B$

$$\det\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \det\begin{bmatrix} B_{11} & B_{12}-\beta B_{11} & B_{13}-\varepsilon B_{11} \\ B_{21} & B_{22}-\beta B_{21} & B_{23}-\varepsilon B_{21} \\ B_{31} & B_{32}-\beta B_{31} & B_{33}-\varepsilon B_{31} \end{bmatrix} \tag{A1.12}$$

for any values of $\beta$ and $\varepsilon$. By choosing

$$\beta = \frac{B_{12}}{B_{11}} = \frac{\psi_{n-2}(E_n)}{\psi_{n-2}(E_{n-1})} \tag{A1.13}$$

and

$$\varepsilon = \frac{B_{13}}{B_{11}} = \frac{\psi_{n-2}(E)}{\psi_{n-2}(E_{n-1})} \tag{A1.14}$$

it is easy to show from (A1.11) and (A1.12) that

$$F \sim G. \tag{A1.15}$$

$G$ is expressed entirely in terms of solutions of the Schrödinger equation for the Hamiltonian $H_{n-2}$. The procedure given above may be extended to write any product of wavefunctions of the form $\psi_m(E_{m+1})\psi_{m+1}(E_{m+2}) \ldots \psi_{n-1}(E_n)\psi_n(E)$ as a determinant involving only the solutions of $H_m$ and their various derivatives. The procedure is straightforward but tedious. The method of proof is indicated below. Assuming that the determinantal relation is true for $m = n-k$, it is possible to prove that it is also true for $m = n-k-1$ by making use of the relations between the wavefunctions given by (54). Since (A1.2), (A1.10), (A1.12) and (A1.15) show that the determinantal relation is true for $m = n-1$ and $m = n-2$, it is then possible to conclude by inductive reasoning that it must be true for any $m$. It is thus possible to write

$$\psi_0(E_1)\psi_1(E_2) \ldots \psi_{n-1}(E_n) \sim \det\begin{bmatrix} \psi_0(E_1) & \psi_0(E_2) & \ldots & \psi_0(E_n) \\ \dot{\psi}_0(E_1) & \dot{\psi}_0(E_2) & \ldots & \dot{\psi}_0(E_n) \\ \vdots & & & \\ \dfrac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\psi_0(E_1) & \dfrac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\psi_0(E_2) & \ldots & \dfrac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\psi_0(E_n) \end{bmatrix}. \tag{A1.16}$$

The elements of the matrix in (A.16) denoted by $D_n$ may be written in the compact form

$$[D_n]_{JK} = \frac{\mathrm{d}^{J-1}}{\mathrm{d}x^{J-1}}\psi_0(E_K). \tag{A1.17}$$

## Appendix 2

In this appendix it is proved that the eigenfunctions of

$$V_n = V_0 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\det D \tag{A2.1}$$

where

$$D_{KJ} = \frac{\mathrm{d}^{K-1}}{\mathrm{d}x^{K-1}}\psi_0(E_J) \tag{A2.2}$$

are given by the elements $(D^{-1})_{in}$, $i=1,2,\ldots,n$. $\psi_0(E_J)$ are the solutions of the Schrödinger equation for the potential $V_0$, i.e.

$$\ddot{\psi}_0(E_J)=[\gamma_J^2+2\mu V_0]\psi_0(E_J). \tag{A2.3}$$

Therefore

$$\ddot{D}_{KJ}=[\gamma_J^2+2\mu V_0]D_{KJ}. \tag{A2.4}$$

Let $B=D^{-1}$. Then

$$BD=DB=I \tag{A2.5}$$

where $I$ is the unit matrix. Hence

$$\frac{d^2}{dx^2}[DB]=\ddot{D}B+2\dot{D}\dot{B}+D\ddot{B}=[0] \tag{A2.6}$$

where [0] is the null matrix and a dot above a matrix indicates that each element of the matrix is differentiated once. Therefore

$$\ddot{B}=B\ddot{D}B-2B\frac{d}{dx}[\dot{D}B]. \tag{A2.7}$$

Using (A2.4) it is easy to show that

$$\ddot{B}_{iJ}=[\gamma_i^2+2\mu V_0]B_{iJ}-2B_{iK}\frac{d}{dx}[\dot{D}B]_{KJ}. \tag{A2.8}$$

It follows from (A2.2) that

$$\dot{D}_{KJ}=D_{K+1,J}[1-\delta_{Kn}]+\delta_{Kn}\dot{D}_{nJ}. \tag{A2.9}$$

Therefore

$$[\dot{D}B]_{KJ}=\delta_{K+1,J}[1-\delta_{Kn}]+\delta_{Kn}[\dot{D}B]_{nJ}. \tag{A2.10}$$

and

$$B_{iK}\frac{d}{dx}[\dot{D}B]_{KJ}=B_{in}[\dot{D}B]_{nJ}. \tag{A2.11}$$

Let $f_{in}$ be the determinant of the cofactor of the element $D_{ni}$. Then

$$B_{in}=\frac{f_{in}}{\det D} \tag{A2.12}$$

and

$$\det D=\sum_i D_{ni}f_{in}. \tag{A2.13}$$

The derivative of a determinant may be evaluated by differentiating the elements in a particular row $i$ of the matrix, evaluating the determinant of this new matrix and then summing over $i$. When the row $i$ $(i\neq n)$ of the matrix $D$ defined by (A2.2) is differentiated, the resulting matrix $D^{(i)}$ has identical elements in the rows $i$ and $i+1$

and hence the determinant of $D^{(i)}$ vanishes. But when the row $n$ of the matrix $D$ is differentiated the resulting matrix $D^{(n)}$ has a non-vanishing determinant. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}\det D=\det D^{(n)}=\det\begin{bmatrix}\psi_0(E_1) & \psi_0(E_2) & \dots & \psi_0(E_n)\\ \dot\psi_0(E_1) & \dot\psi_0(E_2) & \dots & \dot\psi_0(E_n)\\ \ddot\psi_0(E_1) & \ddot\psi_0(E_2) & \dots & \ddot\psi_0(E_n)\\ \vdots & & & \\ \dfrac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}\psi_0(E_1) & \dfrac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}\psi_0(E_2) & \dots & \dfrac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}\psi_0(E_n)\\ \dfrac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\psi_0(E_1) & \dfrac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\psi_0(E_2) & \dots & \dfrac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\psi_0(E_n)\end{bmatrix}. \tag{A2.14}$$

It is easy to show from this expression that

$$\frac{\mathrm{d}}{\mathrm{d}x}\det D=\sum_{J=1}^{n} f_{Jn}\frac{\mathrm{d}}{\mathrm{d}x}D_{nJ}. \tag{A2.15}$$

Using (A2.12) it is then possible to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\det D=[\dot D B]_{nn}. \tag{A2.16}$$

(A2.8), (A2.11) and (A2.16) can be combined to give

$$\ddot B_{in}=\left[\gamma_i^2+2\mu V_0-2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\det D\right)\right]B_{in}. \tag{A2.17}$$

(A2.17) shows that $B_{in}=(D^{-1})_{in}$ is indeed a solution of the Schrödinger equation for the potential $V_n$ given by (A2.1) for the eigenenergy $E_i=-\gamma_i^2/2\mu$.

## Appendix 3

The normalisation of the eigenstates of symmetric reflectionless potentials is discussed in this appendix. The symmetric reflectionless potential with $n$ bound states is given by

$$V_n=-\frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\det D_n \tag{A3.1}$$

where

$$[D_n]_{JK}=\tfrac{1}{2}\gamma_K^{J-1}[\mathrm{e}^{\gamma_K x}+(-1)^{J+K}\,\mathrm{e}^{-\gamma_K x}]. \tag{A3.2}$$

The determinant of $D_n$ may be easily evaluated in the limit $x\to\infty$ to give

$$\lim_{x\to\infty}\det D_n=\left(\prod_{J=1}^{n}\prod_{K>J}^{n}(\gamma_K-\gamma_J)\right)\exp\left(x\sum_K\gamma_K\right). \tag{A3.3}$$

It was shown in appendix 2 that the unnormalised eigenfunctions of $V_n$ are given by

$$\psi_n(E_i)=[D_n^{-1}]_{in}\qquad i=1,2,\dots,n. \tag{A3.4}$$

The determinants involved in the definition of the elements of the inverse matrix can also be evaluated in a similar manner in the limit $x \to \infty$ to give

$$\lim_{x\to\infty} \psi_n(E_i) = 2e^{-\gamma_i x}\left(\prod_{K\neq i}^{n} (\gamma_i - \gamma_K)\right)^{-1}. \tag{A3.5}$$

The same argument can be applied to the eigenfunctions of the potential with $(n-1)$ bound states to give

$$\lim_{x\to\infty} \psi_{n-1}(E_i) = 2e^{-\gamma_i x}\left(\prod_{K\neq i}^{n-1} (\gamma_i - \gamma_K)\right)^{-1}. \tag{A3.6}$$

But the unnormalised eigenfunctions $\psi_n(E_i)$ and $\psi_{n-1}(E_i)$ are related as shown in (54) by

$$\psi_n(E_i) = \lambda_i\left[-\frac{d}{dx} + \left(\frac{d}{dx}\ln \psi_{n-1}(E_n)\right)\right]\psi_{n-1}(E_i) \qquad i \neq n. \tag{A3.7}$$

The constant $\lambda_i$ can be determined by considering the $x \to \infty$ limit of (A3.7). Using the relation

$$\lim_{x\to\infty} \psi_{n-1}(E_n) \sim e^{\gamma_n x} \tag{A3.8}$$

it is possible to show from (A3.5)-(A3.8) that

$$\lambda_i = (\gamma_n^2 - \gamma_i^2)^{-1}. \tag{A3.9}$$

Hence

$$\psi_n(E_i) = \frac{\sqrt{2\mu}}{(\gamma_n^2 - \gamma_i^2)} A_{n-1}^{-}(E_n)\psi_{n-1}(E_i) \tag{A3.10}$$

and

$$\int_{-\infty}^{\infty} \psi_n^2(E_i)\,dx = \frac{2\mu}{(\gamma_n^2 - \gamma_i^2)^2}\int_{-\infty}^{\infty} [A_{n-1}^{-}(E_n)\psi_{n-1}(E_i)]^2\,dx. \tag{A3.11}$$

Partial integration of the integral on the right-hand side of the above equation and use of the expression for $A_{n-1}^{+}$ given by (51) then shows that

$$\int_{-\infty}^{\infty} \psi_n^2(E_i)\,dx = \frac{2\mu}{(\gamma_n^2 - \gamma_i^2)^2}\int_{-\infty}^{\infty} \psi_{n-1}(E_i)A_{n-1}^{+}(E_n)A_{n-1}^{-}(E_n)\psi_{n-1}(E_i)\,dx. \tag{A3.12}$$

But from (50)

$$A_{n-1}^{+}(E_n)A_{n-1}^{-}(E_n) = H_{n-1} - E_n. \tag{A3.13}$$

Therefore

$$\int_{-\infty}^{\infty} \psi_n^2(E_i)\,dx = \frac{1}{(\gamma_n^2 - \gamma_i^2)}\int_{-\infty}^{\infty} \psi_{n-1}^2(E_i)\,dx. \tag{A3.14}$$

When $n = 1$

$$\psi_1(E_1) = \operatorname{sech} \gamma_1 x \tag{A3.15}$$

and so

$$\int_{-\infty}^{\infty} \psi_1^2(E_1)\,dx = 2/\gamma_1. \tag{A3.16}$$

Repeated application of (A3.14) for $i=1$ then shows that

$$\int_{-\infty}^{\infty} \psi_n^2(E_1)\,dx = \frac{2}{\gamma_1}\left(\prod_{K\neq 1}^{n}(\gamma_K^2-\gamma_1^2)\right)^{-1}. \tag{A3.17}$$

When $n=2$, it is easy to show that in terms of the eigenfunctions

$$\psi_2(E_1) = \frac{\sinh \gamma_2 x}{\det D_2} \qquad \psi_2(E_2) = \frac{\cosh \gamma_1 x}{\det D_2} \tag{A3.18}$$

the potential $V_2$ is given by

$$V_2 = -\frac{1}{\mu}\frac{d^2}{dx^2}\ln\det D_2 = -\frac{\gamma_2^2-\gamma_1^2}{\mu}[\gamma_2^2\psi_2^2(E_2)+\gamma_1^2\psi_2^2(E_1)]. \tag{A3.19}$$

But

$$\int_{-\infty}^{\infty}\left(\frac{d^2}{dx^2}\ln\det D_2\right)dx = \frac{d}{dx}\ln\det D_2\Big|_{-\infty}^{\infty} = 2(\gamma_2+\gamma_1) \tag{A3.20}$$

and from (A3.17)

$$\int_{-\infty}^{\infty}\psi_2^2(E_1)\,dx = \frac{2}{\gamma_1}\frac{1}{(\gamma_2^2-\gamma_1^2)}. \tag{A3.21}$$

(A3.19)-(3.21) may be compared to give

$$\int_{-\infty}^{\infty}\psi_2^2(E_2)\,dx = \frac{2}{\gamma_2}\frac{1}{(\gamma_2^2-\gamma_1^2)}. \tag{A3.22}$$

Repeated application of (A3.14) for $i=2$ then shows that

$$\int_{-\infty}^{\infty}\psi_n^2(E_2)\,dx = \frac{2}{\gamma_2}\left(\prod_{K\neq 2}^{n}|\gamma_K^2-\gamma_2^2|\right)^{-1}. \tag{A3.23}$$

Examination of the expression for the elements of $D_n^{-1}$ shows that $[D_n^{-1}]_{J+2,n}$ is obtained from $[D_n^{-1}]_{Jn}$ by the substitution $E_J \leftrightarrow E_{J+2}$ while all the other energies are left unaltered. By using this symmetry property the normalisation factors for all the eigenfunctions can be obtained from those for $\psi_n(E_1)$ and $\psi_n(E_2)$. From (A3.17) and (A3.23), by symmetry

$$\int_{-\infty}^{\infty}\psi_n^2(E_i)\,dx = \frac{2}{\gamma_i}\left(\prod_{K\neq i}^{n}|\gamma_K^2-\gamma_i^2|\right)^{-1}. \tag{A3.24}$$

The normalised eigenfunctions of $V_n$ are therefore given by

$$\tilde{\psi}_n(E_i) = \left(\frac{\gamma_i}{2}\prod_{K\neq i}^{n}|\gamma_K^2-\gamma_i^2|\right)^{1/2}[D_n^{-1}]_{in}. \tag{A3.25}$$

## Appendix 4

In this appendix it is proved that the symmetric reflectionless potential with $n$ bound states may be expressed as a sum over the squares of the normalised eigenfunctions with simple coefficients. In § 5 of the main text it was shown that the symmetric reflectionless potentials constructed using the algebra of supersymmetry is the same

as the potential used by Thacker *et al* (1978). The asymptotic behaviour of the normalised eigenfunctions of $V_n$ given by (A3.25) may be inferred using (A3.4) and (A3.5) to be

$$\lim_{x\to\infty} \tilde{\psi}_n(E_i) = \left(2\gamma_i \prod_{K\neq i} \left|\frac{\gamma_K+\gamma_i}{\gamma_K-\gamma_i}\right|\right)^{1/2} e^{-\gamma_i x}. \tag{A4.1}$$

In terms of the coefficients $C_i$ defined by (74) the normalised eigenfunctions can then be written in the form

$$\tilde{\psi}_n(E_i) = C_i\varphi_i. \tag{A4.2}$$

Then

$$\lim_{x\to\infty} \varphi_i = e^{-\gamma_i x}. \tag{A4.3}$$

The function $\varphi_i$ defined by (A4.2) and (A4.3) is the same function as the $\varphi_i$ defined by Thacker *et al.* It is shown by Thacker *et al* that $\varphi_i$ satisfies

$$\varphi_i e^{\gamma_i x} = 1 - \sum_{J=1}^{n} C_J^2 \varphi_J \frac{e^{-\gamma_J x}}{(\gamma_J+\gamma_i)} \tag{A4.4}$$

and

$$\ddot{\varphi}_i = (\gamma_i^2 + 2\mu V_n)\varphi_i. \tag{A4.5}$$

By differentiating (A4.4) and using (A4.5) it is easy to show that

$$\mu V_n = \sum_J C_J^2 e^{-\gamma_J x}(\dot{\varphi}_J - \gamma_J\varphi_J) = \frac{d}{dx}\left(\sum_J C_J^2 e^{-\gamma_J x}\varphi_J\right). \tag{A4.6}$$

(A4.4) also gives

$$\dot{\varphi}_i - \gamma_i\varphi_i = -2\gamma_i\varphi_i - e^{-\gamma_i x}\sum_J \left(\frac{C_J^2}{(\gamma_J+\gamma_i)}\frac{d}{dx}(e^{-\gamma_J x}\varphi_J)\right). \tag{A4.7}$$

Therefore

$$C_i^2 e^{-\gamma_i x}(\dot{\varphi}_i - \gamma_i\varphi_i) = -2\gamma_i\varphi_i C_i^2 e^{-\gamma_i x} - C_i^2 e^{-2\gamma_i x}\sum_J \frac{C_J^2}{(\gamma_J+\gamma_i)}\frac{d}{dx}(e^{-\gamma_J x}\varphi_J). \tag{A4.8}$$

Use of the expansion

$$e^{-\gamma_i x} = \varphi_i + \sum_J C_J^2\varphi_J \frac{\exp[-(\gamma_J+\gamma_i)x]}{(\gamma_J+\gamma_i)} \tag{A4.9}$$

obtained from (A4.4) in the first term on the right-hand side of (A4.8) enables the writing of (A4.6) in the form

$$\mu V_n = -2\sum_i \gamma_i C_i^2\varphi_i + F(x) \tag{A4.10}$$

where

$$F(x) = -\sum_i\sum_J \frac{C_i^2 C_J^2}{(\gamma_i+\gamma_J)} \times\left(2\gamma_i\varphi_i\varphi_J \exp[(\gamma_i+\gamma_J)x] + \exp(-2\gamma_i x)\frac{d}{dx}(\exp(-\gamma_J x)\varphi_J)\right). \tag{A4.11}$$

The first term in the above equation may be written in the symmetric form

$$\frac{1}{2}\sum_i\sum_j\left(\frac{C_i^2C_j^2}{(\gamma_i+\gamma_j)}(2\gamma_i+2\gamma_j)\varphi_i\varphi_j\exp[-(\gamma_1+\gamma_j)x]\right)=\left(\sum_i C_i^2\exp(-\gamma_i x)\varphi_i\right)^2. \tag{A4.12}$$

The second term of (A4.11) can be simplified using (A4.7). These simplifications lead to the expression

$$F(x)=-\left(\sum_i C_i^2\varphi_i\,e^{-\gamma_i x}\right)^2+\sum_i C_i^2\,e^{-\gamma_i x}(\dot{\varphi}_i+\gamma_i\varphi_i). \tag{A4.13}$$

Therefore

$$\frac{dF}{dx}=-2\left(\sum_i C_i^2\varphi_i\,e^{-\gamma_i x}\right)\frac{d}{dx}\left(\sum_i C_i^2\varphi_i\,e^{-\gamma_i x}\right)+\sum_i C_i^2\,e^{-\gamma_i x}(\dot{\varphi}_i+\gamma_i\varphi_i). \tag{A4.14}$$

Use of (A4.5) and (A4.6) then shows that

$$dF/dx=0. \tag{A4.15}$$

Furthermore, (A4.3) and (A4.13) show that $F(x=\infty)=0$. Since (A4.15) is valid for any value of $x$ it is now possible to conclude that $F(x)=0$ for all values of $x$. In terms of the normalised eigenfunctions defined by (A4.2) it is then possible to write (A4.10) in the form

$$V_n=-\frac{2}{\mu}\sum_i\gamma_i\tilde{\psi}_n^2(E_i). \tag{A4.16}$$

It must be noted that the results derived in appendices 3 and 4 apply only to symmetric reflectionless potentials. The suffix SR used in the main text is omitted in appendices 3 and 4 for simplicity of notation.

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